

# ELLIPTIC CURVES WITH ALL QUADRATIC TWISTS OF POSITIVE RANK

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Imagine you had an elliptic curve  $E/K$  with everywhere good reduction, defined over a number field  $K$  that has no real and an odd number  $r$  of complex places. Then the global root number  $w(E/K)$  is  $(-1)^r = -1$ , and it becomes  $(-1)^{2r} = +1$  over every quadratic extension of  $K$ . As the root number is the sign in the (conjectural) functional equation for the  $L$ -function of  $E$ , the Birch–Swinnerton-Dyer conjecture predicts that the Mordell-Weil rank of  $E$  goes up in *every* quadratic extension of  $K$ . Equivalently, every quadratic twist of  $E/K$  has positive rank, a behaviour that does not occur over  $\mathbb{Q}$  (and would contradict Goldfeld’s “ $1/2$  average rank” conjecture).

These curves do exist<sup>1</sup>. For example, the elliptic curve over  $\mathbb{Q}$

$$E : y^2 = x^3 + \frac{5}{4}x^2 - 2x - 7 \quad (121C1)$$

has discriminant  $-11^4$  and acquires everywhere good reduction over any cubic extension of  $\mathbb{Q}$  which is totally ramified at 11. So one may take  $K = \mathbb{Q}(\zeta_3, \sqrt[3]{11m})$  or  $K = \mathbb{Q}(\sqrt[6]{-11m})$  for any positive  $m$ , coprime to 11. (Those who prefer abelian extensions can take  $E = 1849C1$  and  $K$  to be the degree 6 field inside  $\mathbb{Q}(\zeta_{43})$ .)

It is even easier to construct curves all of whose quadratic twists have root number  $+1$ . For example,

$$E : y^2 = x^3 + x^2 - 12x - \frac{67}{4} \quad (1369E1)$$

has discriminant  $37^3$  and has everywhere good reduction over  $K = \mathbb{Q}(\sqrt[4]{-37})$ . So it has root number  $+1$  over every extension of  $K$ . (Such a field  $K$  exists for every elliptic curve with integral  $j$ -invariant.) In view of the Birch–Swinnerton-Dyer conjecture, we expect  $E$  to have even rank over *every* extension of  $K$ , but it is not at all clear how to prove it for this or any other non-CM elliptic curve<sup>2</sup>.

Let us say that an elliptic curve  $E/K$  is *lawful* if  $w(E/K') = 1$  for every quadratic extension  $K'/K$ , and *chaotic* otherwise. Equivalently,  $E$  is lawful if and only if all of its quadratic twists have the same root number as  $E/K$ . Depending on whether this root number is  $+1$  or  $-1$ , let us call the curve *lawful good* or *lawful evil*. Thus, conjecturally the rank of a lawful evil curve increases in every quadratic extension<sup>3</sup>.

Another way of looking at our first example is that, say, for  $K = \mathbb{Q}(\sqrt[6]{-11})$  the polynomial  $x^3 + \frac{5}{4}x^2 - 2x - 7$  must take *all* values<sup>4</sup> in  $K^*/K^{*2}$ . Generally, one might conjecture that a square-free cubic  $f(x) \in K[x]$  takes “0%”, “50%” or all possible values in  $K^*/K^{*2}$  depending on whether the curve  $y^2 = f(x)$  is lawful good, chaotic or lawful evil over  $K$ .

<sup>1</sup>So Goldfeld’s conjecture fails over number fields. That there are curves all of whose quadratic twists must have positive rank was observed in [6] and is already implicit in [9]. We also get, via Weil restriction, abelian varieties over  $\mathbb{Q}$  all of whose quadratic twists must have positive rank.

<sup>2</sup>Similarly, as Karl Rubin remarked to us, there are fields  $K$  such that  $w(E/K) = 1$  for *every* elliptic curve  $E$  defined over  $\mathbb{Q}$ ; for instance  $\mathbb{Q}(i, \sqrt{17})$  is such a field.

<sup>3</sup>This is also implied by the conjectural finiteness of III, under mild restrictions on  $E$  at  $v|6$ , see [2] Thm. 1.3.

<sup>4</sup>Can one prove (unconjecturally) that such square-free cubics exist, over some  $K$ ? Note that there cannot be a non-constant parametric solution  $(x(t), y(t))$  to  $ty^2 = f(x)$ , for otherwise  $t \mapsto (x(t^2), ty(t^2))$  would be a non-constant map  $\mathbb{P}^1 \rightarrow E$ .

*Classification.* In our examples, we had the unnecessarily strong assumption that the curve has everywhere good reduction. Recall that the global root number  $w(E/K)$  is the product of local root numbers  $w(E/K_v)$  over all places  $v$  of  $K$ . The condition that  $w(E/K') = 1$  for every quadratic extension  $K'/K$  is easily seen to be equivalent to  $w(E/K'_v)$  being 1 for every quadratic extension  $K'_v/K_v$ , for all  $v$ . For instance, this local condition fails for real places but holds for complex places and primes of good reduction for  $E$ .

If  $E$  is an elliptic curve over a local field  $k$ , let us also say that  $E/k$  is *lawful* if  $w(E/k') = 1$  for every quadratic extension  $k'/k$ , and *chaotic* otherwise. Depending on whether  $w(E/k)$  is  $+1$  or  $-1$ , call the curve *lawful good* or *lawful evil*. The reader should be warned that in the local setting lawfulness is not equivalent to the invariance of the root number under quadratic twists.

As mentioned above, a curve over a number field  $K$  is lawful if and only if it is lawful over every completion of  $K$ . Whether it is good or evil is determined by the parity of lawful evil places. For instance, if  $K$  has no real and  $r$  complex places, an elliptic curve  $E/K$  with everywhere good reduction is lawful; it is lawful evil if and only if  $r$  is odd.

It turns out that if  $E/K$  is lawful then  $w(E/F) = w(E/K)^{[F:K]}$  for every extension  $F/K$ . In particular, a lawful evil curve  $E/K$  must acquire points of infinite order over any extension of even degree, while a lawful good curve should have even rank over every extension of  $K$ , as in the example of 1369E1 above. Generally,

**Theorem 1.** *For an elliptic curve  $E$  over a number field  $K$ , the following conditions are equivalent:*

- (1)  $E/K$  is lawful,
- (2)  $E/K_v$  is lawful for all places  $v$  of  $K$ ,
- (3)  $w(E/F) = w(E/K)^{[F:K]}$  for every finite extension  $F/K$ ,
- (4)  $K$  has no real places, and  $E$  acquires everywhere good reduction over an abelian extension of  $K$ ,
- (5)  $K$  has no real places, and for all primes  $p$  and all places  $v \nmid p$  of  $K$ , the action of  $\text{Gal}(\bar{K}_v/K_v)$  on the Tate module  $T_p(E)$  is abelian (“fake CM”)<sup>5</sup>.

This is a corollary of the following local statement.

**Theorem 2.** *For an elliptic curve  $E$  over a local non-Archimedean field  $k$  of characteristic 0, the following conditions are equivalent:*

- (1)  $E/k$  is lawful,
- (2)  $w(E/F) = w(E/k)^{[F:k]}$  for every finite extension  $F/k$ ,
- (3)  $E$  acquires good reduction over an abelian extension of  $k$ ,
- (4) For some (any)  $p$  different from the residue characteristic of  $k$ , the action of  $\text{Gal}(\bar{k}/k)$  on  $T_p(E)$  is abelian.

*Proof.* (2)  $\Rightarrow$  (1) is obvious, and (3)  $\Leftrightarrow$  (4) is a simple consequence of the criterion of Néron–Ogg–Shafarevich in the case of potentially good reduction and the theory of the Tate curve in the potentially multiplicative case. (4)  $\Rightarrow$  (2) is an elementary computation using the formula  $w(\chi)w(\bar{\chi}) = \chi(-1)$  ([8] 3.4.7). As for (1)  $\Rightarrow$  (4), if  $k$  has odd residue characteristic or  $E$  has potentially multiplicative reduction, this follows from the formulae for the local root numbers of Rohrlich [7] and Kobayashi [5].

<sup>5</sup>Recall that  $E$  has CM over  $K$  if and only if the action of the global Galois group  $\text{Gal}(\bar{K}/K)$  on  $T_p(E)$  is abelian; in this case it is obvious that  $E$  has even rank over every extension of  $K$ .

In residue characteristic 2, computing root numbers is a nightmare. Fortunately, Waldspurger has proved this in general on the other side of the local Langlands correspondence: a special or supercuspidal representation  $\pi$  of  $\mathrm{PGL}_2(k)$  has root number  $-1$  after base change to a suitable quadratic extension<sup>6</sup>.  $\square$

An explicit classification of lawful elliptic curves in terms of their  $j$ -invariant has been given by Connell, see [1] Prop. 6. We end with an alternative classification in terms of the Kodaira symbols, which also specifies whether the curve is good or evil. It is easily deduced from the formulae for the local root numbers of Rohrlich, Kobayashi and the authors.

**Classification 3.** Let  $k$  be a local field of characteristic 0, and  $E/k$  an elliptic curve. By [7] Thm. 2,

- If  $E$  has good reduction, then  $E$  is lawful good.
- If  $k \cong \mathbb{C}$ , then  $E$  is lawful evil.
- If  $k \cong \mathbb{R}$  or  $E$  has multiplicative or potentially multiplicative reduction, then  $E$  is chaotic.

Next, suppose  $E/k$  has additive potentially good reduction and  $k$  has odd residue characteristic. Let  $\Delta$  be the minimal discriminant of  $E$  of valuation  $\delta$ , and write  $(\frac{\cdot}{k})$  for the quadratic residue symbol on  $k^*$ . From [7] Thm. 2 and [5] Thm. 1.1,

- $E$  is lawful good if it has type  $I_0^*$  and  $(\frac{-1}{k}) = 1$ , type III, III\* with  $(\frac{-1}{k}) = 1$  and  $(\frac{-2}{k}) = 1$ , or type II, II\*, IV, IV\* with  $\Delta \in k^{*2}$  and  $(\frac{-1}{k})^{\delta/2} = 1$ .
- $E$  is lawful evil if it has type  $I_0^*$  and  $(\frac{-1}{k}) = -1$ , type III, III\* with  $(\frac{-1}{k}) = 1$  and  $(\frac{-2}{k}) = -1$ , or type II, II\*, IV, IV\* with  $\Delta \in k^{*2}$  and  $(\frac{-1}{k})^{\delta/2} = -1$ .
- $E$  is chaotic in all other cases.

(When  $k$  has residue characteristic  $\geq 5$ , Kodaira types II, III, IV,  $I_0^*$ , IV\*, III\*, II\* correspond to  $\delta = 2, 3, 4, 6, 8, 9, 10$ , respectively. Also for  $3 \nmid \delta$ ,  $\Delta \in k^{*2}$  if and only if  $(\frac{-3}{k}) = 1$  in this case, see [5] 1.2.)

Finally, if  $E/k$  has additive potentially good reduction and  $k$  has residue characteristic 2, let  $c_4, c_6$  and  $\Delta$  be the standard invariants of some model of  $E$ , and set  $\gamma(x) = x^8 - 6c_4x^4 - 8c_6x^2 - 3c_4^2 \in k[x]$ . From [3] Prop. 2 and Lemma 3,  $E/k$  is lawful if and only if

- $\sqrt{-3} \in k$  and  $\gamma(x)$  is reducible, or
- $\sqrt{-3} \notin k$ ,  $\sqrt[3]{\Delta} \in k$ , and one of the irreducible factors of  $\gamma(x)$  becomes reducible over  $k(\sqrt{-3})$ .

In this case,  $E$  is lawful good if and only if  $-1$  is a norm from the splitting field of  $\gamma(x)$  to  $k$  ([3] Prop. 4b).

Note from the classification that if  $E/K$  has semistable reduction at places above 2, it becomes lawful over some quadratic extension of  $K$  if and only if it

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<sup>6</sup>If  $E$  has potentially good reduction and the Galois action on  $T_p(E)$  is non-abelian, the dual  $V = (T_p(E) \otimes \mathbb{Q}_p)^*$  is an irreducible representation of the Weil group of  $k$ . Twisting  $V$  by the square root of the cyclotomic character (this does not change the root number) gives a representation that corresponds via local Langlands to a supercuspidal representation  $\pi$  of  $\mathrm{PGL}_2(k)$ . By [9] Prop. 16, there is a quadratic extension  $k'/k$  such that  $w(\pi/k') = -1$ . Because the local Langlands correspondence is known for  $\mathrm{PGL}_2(k)$  (see [4]) and it preserves root numbers and takes restriction to base change, we get  $w(E/k') = -1$ .

has integral  $j$ -invariant. (If  $v|2$  and  $E/K_v$  has additive potentially good reduction,  $E$  stays chaotic in all quadratic extensions of  $K_v$  if and only the inertia group at  $v$  in  $K(E[3])/K$  is  $\mathrm{SL}_2(\mathbb{F}_3)$ .) In this way, it is easy to construct lawful evil curves over imaginary quadratic fields. For example,

$$E : y^2 + xy = x^3 - x^2 - 2x - 1 \quad (49A1)$$

is lawful evil over  $\mathbb{Q}(i)$ , so its rank should go up in every extension of  $\mathbb{Q}(i)$  of even degree.

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